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Paper No. : Classical Mechanics
Module : Hamiltonian principle and Lagrange equations II

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Description of Module	
Subject Name	Physics
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Hamiltonian principle and Lagrange equations II

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2. Open and Closed Orbits.

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Learning Objectives :

- You will learn about the conditions satisfied by a central force to impart a closed or open orbit.
- Learn about the stability criterion of circular orbits.
- You will learn that a small repulsive universe cubic force gives rise to the Perihelion of Mercury.

1. Introduction

A particle in a central–force field may execute bounded or open noncircular motion depending on the nature of the force. If the particle in executing one complete revolution does not return to its original position it signals a deviation from the inverse – square law force however slight. An irregularity in the motion of the planet mercury was observed. It was observed that the perihelion of mercury that the semi-major axis advances at the rate of roughly 574 sec. of arc per century. Calculation of the influence of other planets predicted an advance of approximately 531 sec of arc per century leaving a deficit of about 40 sec of arc per century. Einstein's General Theory of Relativity in post – Newtonian approximation was able to account for this difference of 43 sec of arc per century and became one of the greatest triumphs of General Relativity. In this unit we will discuss the conditions for closed orbits, stability of the orbits and an estimate of the advance of the perihelion of mercury in the presence of deviations from the inverse-square law.

2. Open and Closed Orbits:

We saw earlier that the radial velocity of a particle in a central field is given by

$$\dot{r} = \sqrt{\frac{2}{\mu}(E-U) - \frac{l^2}{\mu^2 r^2}} \qquad (11.1)$$

The velocity will vanish at the roots of $(r_{max}$ and $r_{min})$

$$E - U - \frac{l^2}{\mu^2 r^2} = 0 \qquad (11.2)$$

The motion is therefore confined in the region $r_{min} \le r \le r_{max}$ and these points are the turning points of the motion. For certain value of the potential $U(r)$, there may be only one root in which case the motion is circular. If the motion of the particle is such that the orbit is closed after finite number of round travel between r_{min} and r_{max}, the central potential satisfies certain conditions. If the orbit after finite number of oscillations does not come back to the initial orbit, the orbit is said to be open as shown below:

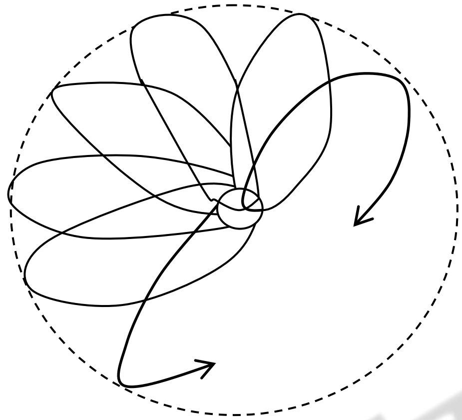

Fig.1

To find the condition for closed orbit, we need to find the change in angle θ after the particle has made one complete oscillation from r_{min} and r_{max} and back; thus

$$\Delta\theta = 2\int_{r_{min}}^{r_{max}} dr$$

Can be evaluated by noting that

$$\frac{d\theta}{dr} = \frac{d\theta}{dt}\frac{dt}{dr} = \frac{\dot{\theta}}{\dot{r}} \quad where\ l = \mu r^2\dot{\theta} \qquad (11.3)$$

Substituting for $\dot{r}$ from (11.1)

$$\Delta\theta = 2\int_{r_{min}}^{r_{max}} \frac{l/_{r^2}}{\sqrt{2\mu\left(E - U - \frac{l^2}{\mu^2 r^2}\right)}} \qquad (11.4)$$

The path will be closed if

$$\Delta\theta = 2\pi\frac{n}{m}$$

Where n and m are integers. Thus after m oscillations $\Delta\theta$ will change by 2π times an integer and would return to its original position i.e. after m periods the particle would have made n complete oscillations.

Example: Show that if the potential $U(r)$ varies as some integer power of radial distance $(r) = kr^{n+1}$ a closed non-circular path will result only if $n = -2,$ or +1.

In terms of the variable $u = \frac{1}{r}$, equation (11.4) can be written as

$$\Delta\theta = -2 \int_{r_{max}}^{r_{min}} \frac{du}{\left(\frac{2\mu E}{l^2} - \frac{2\mu k}{l^2} u^{-n-1} - u^2\right)^{1/2}} \qquad (11.5)$$

The integral (11.5) can be expressed in terms of circular functions provided the radical in the integral (11.5) can be expressed in the form

$$au^2 + bu + c \text{ which is done for } -n-1 = 0, 1 \text{ or } 2 \qquad (11.6)$$

i.e. $n = -1, -2, -3$

$r = -1$ implies constant force and is not of interest, $n = -2$ corresponds to the inverse square for law and $n = -3$ corresponds to inverse-cube force law. The case of $n = -2$ is the familiar case of planetary motion in gravitational field which we considered in detail in the previous unit. The equation of the orbit was found to be $\frac{\alpha}{r} = 1 + \epsilon \cos\theta$ and in this obviously the orbit is closed because as $\theta \to \theta + 2\pi$, the radial is vector returns to its original position.

The case of inverse-cubic force law f is $\frac{1}{r}$ will be treated when we evaluate the advance in the perihelion of mercury. We will find that in this case the orbit does not close on itself. The case of harmonic force $f{\sim}r$ corresponds to $n = 1$ and can be solved in terms of circular functions and has a closed orbit.

3. Orbit in a Linear Force Field

We have the force $f \propto r$ which can be derived from the potential function $U = kr^2$. The equation of the orbit expressed in the variable $u = \frac{1}{r}$ is

$$\theta = -M \int \frac{udu}{\{E - U - Mu^2\}^{1/2}}$$

Where $M = {}^{l}/{\sqrt{2\mu}}$

Putting $U(u) = {k}/{u^2}$

$$\theta = -M \int \frac{udu}{\{Eu^2 - k - M^2u^4\}^{1/2}} \tag{11.7}$$

$$= -M \int \frac{2udu}{\left\{\left(\frac{E^2}{M^2} - k\right) - \left(Mu^2 - \frac{E}{2M}\right)^2\right\}^{1/2}}$$

Putting

$$E'^2 = \frac{E^2}{4l'^2} - k$$

And

$$x = Mu^2 - \frac{E^2}{2l'}$$

$$dx = 2uMdu$$

We get

$$\theta = -\frac{1}{2}\int \frac{dx}{\sqrt{E'^2 - x^2}} = \frac{1}{2}\cos^{-1}\frac{x}{E'}$$

$$x = E' \cos 2\theta \tag{11.8}$$

Substituting back x and E' .

$$\frac{1}{r^2} = \frac{E}{2M^2} + \left(\frac{E^2}{4M^2} - k\right)^{1/2} \frac{1}{M}\cos 2\theta \tag{11.9}$$

4. Stability of Orbits.

If the total energy of the system is equal to the minimum of the effective potential i.e.

$$E = \left(U + \frac{l^2}{2\mu r^2}\right)_{min} = V_{min} \tag{11.10}$$

For any attraction potential U is −ve and the centrifugal potential energy $\frac{l^2}{2\mu r^2}$ can be mode equal to it resulting in a circular orbit. A circular orbit means $r = \varrho$ and $\dot{r}|_{r=\varrho} = 0$ for all t. This is possible if $\left.\frac{\partial V}{\partial r}\right|_{r=\varrho} = 0$. For the orbit to be stale V should have a true minimum at $r = \varrho$ i.e $\left.\frac{\partial^2 V}{\partial r^2}\right|_{r=\varrho} > 0$.

Let us consider a central force

$$f(r) = -\frac{k}{r^n} \tag{11.11}$$

The corresponding potential $U(r)$ is

$$U(r) = -\frac{k}{n-1}\frac{1}{r^{n-1}} \tag{11.12}$$

And

$$V(r) = -\frac{k}{n-1}\frac{1}{r^{n-1}} + \frac{l^2}{2\mu r^2} \tag{11.13}$$

The minimum occurs at

$$\left.\frac{\partial V}{\partial r}\right|_{r=\varrho} = 0 \quad \text{and} \quad \left.\frac{\partial^2 V}{\partial r^2}\right|_{r=\varrho} > 0 \tag{11.14}$$

Implies

$$\frac{k}{\rho^2} - \frac{l^2}{\mu\varrho^2} = 0 \Rightarrow \rho^{n-3} = \frac{\mu k}{l^2} \tag{11.15}$$

And

$$\left.\frac{\partial^2 V}{\partial r^2}\right|_{r=\varrho} = -\frac{nk}{\rho^{n+1}} + \frac{3l^2}{\mu\varrho^4} > 0$$

i.e.

$$\frac{3l^2}{\mu} - \frac{nk}{\rho^{n-3}} > 0 \tag{11.16}$$

Substituting for ρ from equation (11.15)

$$(3-n)\frac{l^2}{\mu} > 0 \tag{11.17}$$

The orbit is stable for $n < 3$

For a general central force field

$$f(r) = -\mu g(r) = -\frac{\partial U}{\partial r} \tag{11.18}$$

The Lagrange's equation of motion is

$$\ddot{r} = r\dot{\theta}^2 = -g(r) \tag{11.18}$$

Consider the particle to be in a circular orbit of radius ρ. To examine the stability, let the radius is perturbed $r \to \rho + x$ where x is small. Now $\ddot{r} = \ddot{x}$ and equation (11.18) by substituting for $\dot{\theta} = \frac{l}{\mu r^2}$ becomes

$$\ddot{x} - \frac{l^2}{\mu^3\rho^3\left(1+\frac{x}{\varrho}\right)^3} = -g(\rho + x) = -g(\rho) - g'(\rho) + \ \ldots.$$

and expanding $\left(1+\frac{x}{\varrho}\right)^{-3} = 1 - \frac{3x}{\varrho} + \ \ldots.$

we get

$$\ddot{x} - \frac{l^2}{\mu^2\rho^3}\left(1+\frac{x}{\varrho}\right) \simeq -\left(g(\rho) + xg'(\varrho)\right) \tag{11.19}$$

For a circular orbit $\ddot{r} = 0$ and $g(\varrho) = \frac{l^2}{\mu^2\rho^3}$ and

$$\ddot{x} + \left\{\frac{2g(\varrho)}{\varrho} + g'(\varrho)\right\}x \simeq 0 \tag{11.20}$$

This has a solution given by

$$x(t) = Ae^{-i\omega t} + Be^{i\omega t} \tag{11.21}$$

Where A and B are constants and

$$\omega^2 = \frac{3g(\rho)}{\rho} + g'(\rho) \tag{11.22}$$

For the stability of the orbit, it is required that ω is imaginary x will increase exponentially making the ρ orbit to be unstable. Therefore the stability of the orbit is ensured provided

$$\frac{3g(\rho)}{\rho} + g'(\varrho) > 0$$

$$\frac{g'(\varrho)}{g} + \frac{3}{\rho} > 0 \tag{11.23}$$

5. Perihelion of Mercury

We mentioned in the introduction that the perihelion of Mercury that is semi-major axis was found to advance at the rate of roughly 574 sec. of arc per century. This would happen if the law of force deviated to the inverse square law. We will assume the presence of a small $\frac{1}{r^3}$ force in addition to the inverse square law to understand the advance in the perihelion. This small $\frac{1}{r^3}$ force could for example arise due to the pressure of other gravitating bodies i.e. other planets, comets, etc. apart from the sun which provides the major inverse square force. We take the potential $U(r)$ to be

$$U(r) = -\frac{k}{r} + \frac{\beta}{r^2} = -ku + \beta u^2 \tag{11.24}$$

And β is assumed to be small.

The orbit equation is given by

$$\theta = -M \int \frac{du}{\sqrt{E - U - M^2u^2}} \tag{11.25}$$

Substituting for U from (10.24) and writing

$$M_\beta^2 = M^2 + \beta, \quad \text{we get}$$

$$\theta = -M \int \frac{du}{\{E + ku - (M^2 + \beta)u^2\}^{1/2}}$$

$$\frac{M_\beta}{M}\theta = -M_\beta \int \frac{du}{\{E + ku - M_\beta^2 u^2\}^{1/2}}$$

$$= -M_\beta \int \frac{du}{\left\{\left(E + \frac{k^2}{4M_\beta^2}\right) - M_\beta^2 u^2 - \frac{k^2}{4M_\beta^2} + ku\right\}^{1/2}}$$

$$= -M_\beta \int \frac{du}{\left\{E'^2 - \left(M_\beta u - \frac{k}{2M_\beta}\right)^2\right\}^{1/2}}$$

Which can be written as

$$\alpha\theta = -\int \frac{dx}{\{E'^2 - x^2\}^{1/2}} \tag{11.26}$$

Where

$$\alpha = \frac{M_\beta}{M}; (E')^2 = E + \frac{k^2}{4M_\beta^2} \quad \text{and}$$

$$x = M_\beta u - \frac{k}{2M_\beta} \tag{11.27}$$

The solution of (10.26) is given by

$$\frac{1}{r} = \frac{1 + e\cos\alpha\theta}{2M_\beta^2 / k}$$

Where the eccentricity e is given by

$$e = \sqrt{1 + \frac{4M_\beta^2 E}{k^2}} \tag{11.28}$$

Now for$E < 0$, the eccentricity $e < 0$, the orbit is an ellipse, but it is not a closed ellipse because as $\theta \to \theta + 2\pi$ the particle does not come back to its original position. In the presence of a small respective inverse-cubic force.

$$\theta \to \alpha\theta = \theta\left(\frac{M^2 + \beta}{M^2}\right)^{1/2} \simeq \theta + \frac{1}{2}\frac{\beta}{M^2}\theta$$

$$\backsimeq \theta + \frac{\mu\beta}{l^2}\theta \qquad (11.29)$$

Thus the perihelion advance or the shift in the semi-major axis is proportional to the strength of the inverse cubic repulsive force and the orbit is a '**precessing ellipse'** shown in Fig. 1.

6. Summary

- A particle moving in a force field where the force is either inverse-square or a linear harmonic force has a closed orbit.
- The inverse cubic force law results in an open precessing elliptical orbit.
- The circular orbit of a particle moving in a general central force field is stable provided

$$\frac{d}{dr}(\log g(r))\Big|_{r=\varrho}$$

where ρ is the radius of the circular orbit.

- The perihelion of mercury arises because of the presence of a small repulsive inverse cubic force.